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MAGNETOHYDRODYNAMIC CHANNEL FLOW HEAT TRANSFER FOR TEMPERATURE BOUNDARY CONDITION OF THE THIRD KIND

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Abstract—To investigate the influence of the temperature boundary condition of the third kind on the magnetohydrodynamic heat transfer in the thermal entrance region of a flat channel, the energy equation is solved by applying the Galerkin–Kantorowich method of variational calculus. The Hartmann velocity profile is assumed. The heat generation within the fluid is neglected. It is concluded that there can be a significant influence of the Biot number on the local Nusselt number. Representative results are depicted in tables.

NOMENCLATURE

[ ], matrix;
{ }, column vector;
A, channel cross section;
B, magnetic induction;
Bi, Biot number, equation (4);
D, a matrix, equation (8);
F, a vector, equation (5);
Ha, Hartmann number, equation (4);
Nu, Nusselt number, equation (15);
Pe, Peclet number, equation (4);
R, a vector, equation (11);
T, temperature;
W, a matrix, equation (8);
c, half channel height;
c_p, specific heat at constant pressure;
e, eigenvalue, equation (16);
f, a function, equation (5);
k, overall heat-transfer coefficient;
s, characteristic value, equation (5);
v, velocity;
x, y, z, cartesian coordinate.

Greek symbols
η, dynamic viscosity;
λ, thermal conductivity;
μ, magnetic permeability;
ρ, mass density;
σ, electrical conductivity.

Subscripts
a, ambient;
i, j, running index;
m, mean value;
w, wall;
x, y, z, cartesian coordinate direction;
0, prescribed value.

Superscripts
$\bar{}$, dimensionless quantity, equation (4);
$'$, $(d/d\bar{x})$.

1. INTRODUCTION

IN THE previous analyses on the magnetohydrodynamic (MHD) laminar forced convection heat transfer in the thermal entrance region of a channel, either the boundary condition of the first kind characterized by the prescribed wall temperature [1–4] or the boundary condition of the second kind expressed by the prescribed wall heat flux is assumed [5–9]. A more realistic condition in many applications, however, will be the temperature boundary condition of the third kind: the local wall heat flux is a linear function of the local wall temperature. This situation is encountered in the heat-transfer process, where the radiative heat transfer, describable in terms of Newton's law of cooling, occurs at the channel wall and is, to the author's knowledge for an MHD channel, not reported in the literature.

The objective of the present paper is to investigate the MHD laminar forced convection heat transfer in the thermal entrance region of a flat channel for the temperature boundary condition of the third kind. Assuming constant fluid properties and fully developed Hartmann flow, the energy equation is solved by employing the Galerkin–Kantorowich method of variational calculus. Since the main concern of this analysis is with the influence of the finite wall thermal resistance on the heat transfer, the axial heat conduction and the heat generation within the fluid are not considered. Their influence is already discussed in [9–11].

2. ANALYSIS

Consider steady, fully developed laminar MHD Hartmann flow in a flat channel (Fig. 1). For the region $x \geqslant 0$, where a constant ambient temperature is maintained, a uniform magnetic field is imposed in z direction and the electric current is allowed to flow in y direction. Assuming constant fluid properties and neglecting the heat conduction in flow direction and the heat generation within the fluid, the laminar forced heat convection subject to boundary condition of the

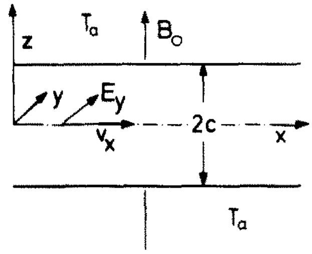

FIG. 1. MHD channel under investigation.

third kind for temperature can be described by the equations [12]:

$$\bar{v}_x = Ha[\cosh(Ha) - \cosh(Ha\bar{z})]/[Ha\cosh(Ha) - \sinh(Ha)], \quad (1)$$

$$L(\bar{T}) \equiv Pe\bar{v}_x(\partial\bar{T}/\partial\bar{x}) - \partial^2\bar{T}/\partial\bar{z}^2 = 0, \quad (2)$$

$$\begin{aligned} &\bar{x} = 0:\ T = T_0,\ \bar{T} = 1,\\ &\bar{x} \to \infty:\ T \to T_a,\ \bar{T} \to 0,\\ &\bar{z} = 0:\ (\partial T/\partial z) = 0,\ (\partial\bar{T}/\partial\bar{z}) = 0,\\ &|\bar{z}| = 1:\ \lambda(\partial T/\partial z) + k(T - T_a) = 0,\\ &\qquad (\partial\bar{T}/\partial\bar{z}) + Bi\bar{T} = 0, \end{aligned} \quad (3)$$

where k is the overall heat-transfer coefficient based on the wall thermal resistance and on the ambient side surface resistance. The dimensionless quantities employed are

$$\begin{aligned} &\bar{x} = x/c,\ \bar{z} = z/c,\\ &\bar{v}_x = v_x/v_{x,m},\ v_{x,m} = (1/A)\int_A v_x\,dA,\\ &\bar{T} = (T - T_a)/(T_0 - T_a),\ Pe = v_{x,m}cc_p\rho/\lambda,\\ &Bi = ck/\lambda,\ Ha = cB_0\sqrt{(\sigma/\eta)}. \end{aligned} \quad (4)$$

To solve the energy equation (2), the Galerkin–Kantorowich method of variational calculus is employed, which allows to reduce a partial differential equation to an ordinary one [13]. Let the approximate temperature field be

$$\bar{T} = \sum_j f_j \cos(s_j\bar{z})$$

$$\{F\} = \{f_1(\bar{x}), f_2(\bar{x}), \ldots, f_N(\bar{x})\},$$

$$\cos(s_j) - (s_j/Bi)\sin(s_j) = 0, \quad (5)$$

where the characteristic values s_j are to be determined, so that the boundary condition is satisfied. Taking the energy equation (2) with the natural boundary condition (3) as the Euler equation of the variational formulation, one may solve

$$\int_0^1 L(\bar{T})\,\partial\bar{T}\,d\bar{z} = 0 \quad (6)$$

to evaluate the unknown functions $f_j(x)$.

With

$$\delta\bar{T} = \sum_j (\partial\bar{T}/\partial f_j)\delta f_j, \quad (7)$$

from equations (2) and (6), one can derive a system of ordinary differential equations for $f_j(\bar{x})$ as follows (for details see Appendix)

$$Pe[D]\{F'\} = [W]\{F\}. \quad (8)$$

To calculate the value of the unknown functions at the channel entrance, one may use the condition

$$\int_0^1 g(\bar{z})[\bar{T}(\bar{x} = 0) - \bar{T}_0]^2\,d\bar{z} \to \min. \quad (9)$$

Setting the weighting function as

$$g(\bar{z}) = \bar{v}_x, \quad (10)$$

from equations (5) and (9), one can deduce a system of algebraic equations for $f_j(\bar{x} = 0)$ as follows (for details see Appendix)

$$[D]\{F(\bar{x} = 0)\} = \{R\}. \quad (11)$$

The characteristic quantities describing the heat transfer at the channel walls are

$$(\partial\bar{T}/\partial\bar{z})_w = -\sum_j f_j s_j \sin(s_j), \quad (12)$$

$$\bar{T} = \sum_j f_j \cos(s_j), \quad (13)$$

$$\bar{T}_m = \int_0^1 \bar{T}\bar{v}_x\,d\bar{z} = \sum_j f_j R(j), \quad (14)$$

$$Nu = 4(\partial\bar{T}/\partial\bar{z})_w/(\bar{T}_w - \bar{T}_m). \quad (15)$$

Table 1. Local Nusselt numbers for $Ha \to \infty$ and different values of Bi

$\bar{x}/Pe$	$Bi = 0.01$	0.1	1.0	10.0	100.0	∞	∞
			$N = 20$ in (5)				$N = 50$ in (16)
0.0001	286.45	286.28	284.54	268.01	187.64	143.22	222.91
0.0002	233.31	233.19	232.00	220.66	163.08	128.47	161.95
0.0005	160.04	159.95	159.13	151.61	118.39	98.703	103.54
0.001	115.24	115.16	114.39	107.71	84.672	73.631	74.006
0.002	82.525	82.446	81.679	75.428	59.001	53.129	53.144
0.005	53.471	53.392	52.627	47.044	37.152	34.671	34.683
0.01	38.885	38.805	38.044	33.089	26.714	25.432	25.438
0.02	28.648	28.567	27.811	23.565	19.642	18.984	18.987
0.05	19.760	19.678	18.936	15.690	13.765	13.494	13.498
0.1	15.564	15.481	15.221	12.279	11.227	11.092	11.095
0.2	13.097	13.017	12.359	10.630	10.101	10.037	10.039
0.5	12.043	11.972	11.427	10.264	9.9154	9.8699	9.8701
1.0	11.992	11.923	11.395	10.262	9.9148	9.8696	9.8696

Table 2. Local Nusselt numbers for different values of Ha and Bi

$\bar{x}/Pe$	0.01	0.1	Bi 1.0	10.0	100.0	∞
			$Ha = 0$			
0.0001	82.323	82.282	81.888	78.997	74.708	72.252
0.0002	64.104	64.060	63.636	60.551	55.212	54.051
0.0005	47.302	47.260	46.856	44.193	40.109	39.035
0.001	37.571	37.529	37.131	34.727	31.741	31.054
0.002	29.893	29.850	29.459	27.320	25.148	24.706
0.005	22.193	22.151	21.769	19.988	18.607	18.362
0.01	17.818	17.775	17.403	15.888	14.930	14.774
0.02	14.427	14.385	14.024	12.764	12.112	12.014
0.05	11.175	11.133	10.795	9.8494	9.4721	9.4197
0.1	9.5250	9.4846	9.1739	8.4562	8.2213	8.1904
0.2	8.5689	8.5326	8.2716	7.7816	7.6483	7.6314
0.5	8.2397	8.2096	8.0040	7.6503	7.5535	7.5410
1.0	8.2319	8.2023	8.0000	7.6499	7.5532	7.5410
			$Ha = 4$			
0.0001	102.09	102.06	101.73	99.270	93.517	87.264
0.0002	77.563	77.520	77.102	74.002	68.946	66.948
0.0005	56.384	56.340	55.919	52.958	47.763	46.403
0.001	44.638	44.595	44.181	41.498	37.606	36.614
0.002	35.341	35.297	34.885	32.452	29.578	28.940
0.005	26.034	25.990	25.582	23.497	21.625	21.270
0.01	20.754	20.709	20.305	18.489	17.170	16.944
0.02	16.666	16.620	16.222	14.673	13.761	13.618
0.05	12.745	12.698	12.311	11.104	10.568	10.491
0.1	10.742	10.696	10.328	9.3812	9.0423	8.9970
0.2	9.5536	9.5103	9.1869	8.5157	8.3216	8.2968
0.5	9.1104	9.0735	8.8123	8.3307	8.1935	8.1756
1.0	9.0973	9.0610	8.8049	8.3297	8.1931	8.1752
			$Ha = 10$			
0.0001	133.69	133.66	133.35	130.77	117.38	104.17
0.0002	100.77	100.74	100.38	97.645	91.236	85.302
0.0005	70.804	70.759	70.319	67.051	60.992	59.069
0.001	55.566	55.520	55.083	52.017	46.695	45.199
0.002	43.692	43.646	43.208	40.374	36.326	35.310
0.005	31.831	31.783	31.340	28.820	26.097	25.528
0.01	25.122	25.073	24.624	22.362	20.398	20.036
0.02	19.940	19.889	19.435	17.441	16.060	15.831
0.05	14.976	14.923	14.465	12.841	12.014	11.893
0.1	12.433	12.378	11.928	10.609	10.085	10.013
0.2	10.894	10.841	10.426	9.4597	9.1607	9.1226
0.5	10.276	10.229	9.8842	9.1951	8.9903	8.9634
1.0	10.254	10.207	9.8701	9.1934	8.9897	8.9629
			$Ha = 20$			
0.0001	163.42	163.38	162.96	159.09	135.15	115.80
0.0002	125.41	125.37	125.03	122.25	110.62	99.382
0.0005	86.430	86.380	85.946	82.625	75.574	71.761
0.001	66.629	66.582	66.119	62.709	56.086	53.888
0.002	51.840	51.791	51.321	48.084	42.717	41.203
0.005	37.211	37.159	36.675	33.702	29.976	29.125
0.01	28.987	28.933	28.435	25.698	22.971	22.434
0.02	22.673	22.616	22.103	19.636	17.709	17.376
0.05	16.675	16.615	16.085	14.036	12.899	12.730
0.1	13.629	13.566	13.037	11.364	10.661	10.566
0.2	11.781	11.719	11.224	9.9986	9.6115	9.5628
0.5	11.011	10.956	10.541	9.6812	9.4225	9.3886
1.0	10.979	10.925	10.520	9.6791	9.4219	9.3882

3. RESULTS

To explore the influence of Biot number on the MHD channel flow heat transfer, equations (8) and (11) were solved by employing the standard Crank–Nicolson procedure for the different values of Hartmann number limited by the range $0 \leqslant Ha \leqslant \infty$.

To assess the accuracy of the present results, the special case of slug flow ($Ha \to \infty$) was analyzed for several number of terms in temperature approximation (5). The local Nusselt numbers according to equation (15) for slug flow were compared with the results derived from the exact solution

$$\bar{T} = 2\sum_j [\sin e_j/(e_j + \sin e_j \cos e_j)] \times \cos(e_j \bar{z}) \exp(-e_j \bar{x}/Pe), \quad (16)$$

$$(e_j/Bi) \sin e_j = \cos e_j.$$

It was found that the accuracy of the results depends strongly upon the number of terms considered in equations (5) and (16): closer a location to the channel entrance, larger the number of terms needed to describe a good approximation for this location. From Table 1, for the special case of $Bi \to \infty$, one can learn that the present local Nusselt numbers based on the first twenty terms in equation (5) agree well with the results based on the first fifty eigenvalues e_j in equation (16) for $(\bar{x}/Pe) > 0.0005$. This accuracy seems to be reasonable for the practical engineering purposes. Consequently, all other results are obtained by including the first twenty terms in the approximate solution (5).

In Tables 1 and 2, the local Nusselt numbers based on the first twenty terms in equation (5) are presented for a few arbitrarily selected values of Ha and Bi including the special cases

$Ha \to 0$: Hagen–Poiseuille flow,

$Ha \to \infty$: slug flow,

$Bi \to \infty$: prescribed wall temperature.

The other special case of $Bi \to 0$ cannot be treated with the approximation (5), since the wall heat flux tends to zero. For this particular case of prescribed wall heat flux, one has to formulate other approximation, for instance, as in [9].

From the results obtained, one can conclude, that there can be a substantial influence of the finite wall thermal resistance on the MHD channel flow heat transfer, for instance one has

$$\frac{Nu(Ha = 4,\ \bar{x}/Pe = 0.001,\ Bi = 0.01)}{Nu(Ha = 4,\ \bar{x}/Pe = 0.001,\ Bi = 100)} = 1.19,$$

$$\frac{Nu(Ha = 4,\ \bar{x}/Pe \to \infty,\ Bi = 0.01)}{Nu(Ha = 4,\ \bar{x}/Pe \to \infty,\ Bi = 100)} = 1.11,$$

and the analysis of the problem by neglecting this effect may result in considerable error in the solutions representing the actual physical conditions.

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APPENDIX

The elements of the vectors and matrices occuring in equations (8) and (11) are listed below.

$$i = 1, 2, \ldots, N.$$

$$j = 1, 2, \ldots, N.$$

$$a_1 = s_j - s_i, \quad a_2 = s_j + s_i.$$

$$C_1 = \cos(s_i), \qquad S_1 = \sin(s_i),$$

$$P_1 = \cos(a_1), \qquad Q_1 = \sin(a_1),$$

$$P_2 = \cos(a_2), \qquad Q_2 = \sin(a_2),$$

$$W(i,j) = -(s_j^2/2)(Q_1/a_1 + Q_2/a_2).$$

$$Ha \to \infty$$

$$R(i) = S_1/s_i, \quad D(i,j) = (Q_1/a_1 + Q_2/a_2)/2.$$

$$Ha \to 0$$

$$R(i) = (3/2)[-2C_1/s_i^2 + 2S_1/s_i^3],$$

$$D(i,j) = (3/2)[-P_1/a_1^2 + Q_1/a_1^3 - P_2/a_2^2 + Q_2/a_2^3].$$

$$0 < Ha < \infty$$

$$H_1 = \cosh(Ha), H_2 = \sinh(Ha), H_0 = Ha/(HaH_1 - H_2),$$

$$R(i) = H_0[H_1 S_1/s_i - (HaH_2 C_1 + H_1 S_1 s_i)/(Ha^2 + s_i^2)],$$

$$D(i,j) = (H_0/2)[H_1(Q_1/a_1 + Q_2/a_2) - (HaH_2 P_1 + a_1 H_1 Q_1)/(Ha^2 + a_1^2) - (HaH_2 P_2 + a_2 H_1 Q_2)/(Ha^2 + a_2^2)].$$

TRANSFERT DE CHALEUR DANS UN CANAL MAGNETOHYDRODYNAMIQUE AVEC CONDITIONS AUX LIMITES DE TROISIEME ESPECE

Résumé—Afin d'étudier l'influence des conditions aux limites, de troisième espèce, de température sur le transfert de chaleur dans la région d'établissement thermique d'un canal magnétohydrodynamique bidimensionnel, on résout l'équation d'énergie à l'aide de la méthode de Galerkin–Kantorowich du calcul des variations. On suppose un profil de vitesse de Hartman et on néglige la production de chaleur à l'intérieur du fluide. On est amené à conclure que le nombre de Biot peut avoir une influence non négligeable sur le nombre de Nusselt. Les résultats sont présentés sous forme de tables.

МАГНИТОГИДРОДИНАМИЧЕСКИЙ ТЕПЛОПЕРЕНОС ПРИ ТЕЧЕНИИ В КАНАЛЕ С ТЕПЛОВЫМИ ГРАНИЧНЫМИ УСЛОВИЯМИ ТРЕТЬЕГО РОДА

Аннотация — Для исследования влияния теплового граничного условия третьего рода на магнитогидродинамический теплоперенос в начальном тепловом участке плоского канала решалось уравнение энергии с помощью вариационного метода Галеркина–Канторовича. Принято допущение о Хартмановском виде профиля скорости и об отсутствии тепловыделения в жидкости. Сделан вывод, что число Био может оказывать существенное влияние на локальное число Нуссельта. Результаты представлены в таблицах.